

On the Nearest Neighbor Algorithm for Mean Field Traveling Salesman Problem

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Abstract

In this work we consider the mean field traveling salesman problem, where the intercity distances are taken to be i.i.d. with some distribution F . This paper focus on the *nearest neighbor tour* which is to move to the nearest non-visited city and we show that under some conditions on F , the total length of the nearest neighbor tour, asymptotically almost surely scales as $\log n$. Similar result is known for Euclidean TSP and nearest neighbor tour. We further derive the limiting behavior of the total length of the nearest neighbor tour for a general distribution function F with certain assumptions and show that its asymptotic properties are determined by the scaling properties of the density of F at 0.

1 Introduction

The traveling salesman problem (TSP) is a very well known combinatorial optimization problem. The aim is to find the shortest tour, connecting a number of cities visited by a traveling salesman on his sales route, such that he visits each city exactly once and finally returns to the starting city. Formally, we are given a set $\{c_1, c_2, \dots, c_n\}$ of *cities* and for each pair $\{c_i, c_j\}$ of distinct cities, a distance $d(c_i, c_j)$. The goal is to find a permutation π of the cities that minimizes the quantity

$$\sum_{i=1}^n d(c_{\pi(i)}, c_{\pi(i+1)}) \quad (1.1)$$

where $\pi(n+1) = 1$. This quantity is called the *tour length*, since it is the total distance traveled by the salesman. We shall concentrate in this chapter on the *symmetric* TSP, in which the distances satisfy

$$d(c_i, c_j) = d(c_j, c_i) \quad \text{for } 1 \leq i, j \leq n.$$

There are several randomized versions of this problem where the distances are taken to be random. In particular the one which attracted considerable attention among mathematicians and computer scientists is known as the *Euclidean TSP*, in which the n cities are randomly distributed in a d -dimensional hypercube and the distances between cities are given by the Euclidean metric and are thus random. The other random TSP, which has been of interest within the statistical physics community is the *mean field TSP*. Here the distances between

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pairs of cities, i.e., $d(c_i, c_j)$ are taken as independent random variables with a given distribution F . Note that in this case, the geometric structure may break since the triangle inequality may not necessarily hold with probability one. In fact we can not quite say that the numbers $d(c_i, c_j)$ really represent distances under any metric. This although seems artificial, but such models are of interest in statistical physics literature.

It is well known in algorithm literature [5] that TSP in general is a *NP-Complete* problem. So there are several approximate algorithms which tries to approximate the optimal tour with polynomial running time. Among them, one of the simplest is the *Nearest Neighbor (NN) Algorithm* [2], which is also known as *the next best method* [3]. It was one of the first algorithms used to determine an approximate solution to the traveling salesman problem. The algorithm starts with a tour containing a randomly chosen city and then always adds the nearest not yet visited city to the last city in the tour. The algorithm terminates when every city has been added to the tour. In the NN algorithm, a tour is constructed as follows:

Step-0: Input graph G with a linear ordering of its vertices say

$$V := \{c_1, c_2, \dots, c_n\}.$$

Let $Tour \leftarrow \{c_1\}$ and $c_{\pi(1)} = c_1$.

Step-1: Write $Tour \leftarrow \{c_{\pi(1)}, c_{\pi(2)}, \dots, c_{\pi(i)}\}$. Choose $c_{\pi(i+1)}$ to be the city c_j that minimizes

$$\{d(c_{\pi(i)}, c_j) : j \neq \pi(k), 1 \leq k \leq i\}.$$

Update $Tour$ as

$$Tour \leftarrow Tour \cup \{c_{\pi(i+1)}\}.$$

Step-2: Go to Step-1 unless $V \setminus Tour = \emptyset$.

Step-3: Stop with output $Tour$ as the NN tour with starting city c_1 .

For the convenience, when there are ties in Step-1, we assume that they can be broken arbitrarily. The NN algorithm can be improved by repeating the algorithm for each possible starting city and then take the minimum solution among them [3]. It is known that, for TSP on n cities, the running time for NN algorithm is $O(n^2)$ [4, 6].

Consider the mean field TSP on a set of n cities $\{c_1, c_2, \dots, c_n\}$. Denote the distance $d(c_i, c_j)$ by L_{ij} . Since the NN algorithm is to move to the nearest non-visited city, therefore starting from c_1 , by using this algorithm we need to find the nearest city to it. We call it v_2 . In this way, we need to find

$$\min \{L_{12}, L_{13}, \dots, L_{1n}\}$$

Then from city v_2 we find the nearest city to that and call it v_3 . Here we need to find

$$\min \{L_{v_2 u} | u \in \{2, 3, \dots, n\} \text{ and } u \neq v_2\}.$$

We continue the algorithm till all n cities have been visited. Then from there we go back to first visited city which is c_1 .

Define T_n^{NN} to be the length of NN tour among n cities in the TSP, then

$$T_n^{NN} = \sum_{i=1}^n L_{v_i v_{i+1}}, \quad v_1 = 1 = v_{n+1} \quad (1.2)$$

The performance of nearest neighbor algorithm has been studied for the Euclidean TSP to compare the length of NN tour with the optimal one. Let T_n^{opt} be the length of the optimal tour

and $\lceil x \rceil$ denote the smallest integer greater than or equal to x . Rosenkrantz, Stearns and Lewis [6] measured the closeness of a tour by the ratio of the obtained tour length, to the optimal tour length. They proved that for a Euclidean TSP with n cities,

$$\frac{T_n^{NN}}{T_n^{opt}} \leq \frac{1}{2} \lceil \log_2 n \rceil + \frac{1}{2}.$$

They also showed that for each $m > 3$, there exists a traveling salesman graph with $n = 2^m - 1$ nodes such that

$$\frac{T_n^{NN}}{T_n^{opt}} > \frac{1}{3} \log_2(n+1) + \frac{4}{9}.$$

One of the famous mathematical results for the Euclidean TSP is Beardwood-Halton-Hammersley theorem which studies the large sample behavior of the length of shortest tour in TSP. Let the cities are independently and uniformly distributed on $[0, 1]^d$. Beardwood, Halton and Hammersley [1] showed that there is a constant $0 < \beta_{TSP}(d) < \infty$ such that with probability one

$$\frac{T_n^{opt}}{n^{\frac{d-1}{d}}} \longrightarrow \beta_{TSP}(d)$$

They also proved that for nonuniform random samples, there is an universal constant $\beta_{TSP}(d)$ such that

$$\frac{T_n^{opt}}{n^{\frac{d-1}{d}}} \longrightarrow \beta_{TSP}(d) \int_{\mathbb{R}^d} f(x)^{(d-1)/d} dx \quad a.s.$$

where $f(x)$ is the density of the absolutely continuous part of the distribution of cities with a compact support.

Asymptotic results in the mean field TSP is studied by Wästlund [7]. Let independent random variables L_{ij} 's be from a fixed distribution on the nonnegative real numbers. Suppose as $t \rightarrow 0^+$

$$\frac{\mathbb{P}(L_{ij} < t)}{t} \longrightarrow 1$$

He proved that for large n ,

$$T_n^{opt} \xrightarrow{\mathbb{P}} \frac{1}{2} \int_0^\infty h(x) dx \quad (1.3)$$

where h as a function of x is implicitly defined through the equation

$$\left(1 + \frac{x}{2}\right) e^{-x} + \left(1 + \frac{h(x)}{2}\right) e^{-h(x)} = 1$$

Although there seems to be no simple expression for this limit in terms of known mathematical constants, but it can be evaluated numerically to 2.041548.

In this paper we study the limiting behavior of the total length of the tour, obtained by NN algorithm for the mean field TSP. Our motivation is similar to that of [6]. We would like to compare the apparent “loss” (that is, more distance to be traversed) accrued by using the NN algorithm with respect to the optimal solution. But because of (1.3), it is enough to consider the limiting behavior of T_n^{NN} . We show if F , the distribution of the distance between cities, has a density which is continuous at 0 with $F'(0+) > 0$, then the total length of the NN tour for mean field TSP scales as $\log n$. This parallels the conclusions drawn in [6] for Euclidean TSP. Moreover we also consider general distribution function F with non-negative support and show that the asymptotic behaviors for T_n^{NN} depend on the limiting properties of the density near 0.

2 Main results

We will assume that the mean and the variance of F are finite and F has a density f . Our first theorem shows that T_n^{NN} is “close” to its expected value.

Theorem 2.1. *Assume that as $t \rightarrow 0+$, $\frac{f(t)}{t^\alpha} \rightarrow C$, where $C \in (0, \infty)$ is constant and $-1 < \alpha < 1$. Then as $n \rightarrow \infty$,*

$$\{T_n^{NN} - \mathbb{E}[T_n^{NN}]\}_{n \geq 1} \text{ converges in } \mathcal{L}_2 \quad (2.1)$$

The next three results consider three cases of the behavior of f near 0. Theorem 2.2 covers the case when f near zero converges to a constant. In this case, T_n^{NN} scales as constant times $\log n$. Theorem 2.3 and Theorem 2.4 consider the cases when $\lim_{t \rightarrow 0} f(t)$ is zero and infinity respectively. We use the notation $a_n \sim b_n$ to denote a_n is asymptotically equal to b_n , that is, $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1$.

Theorem 2.2. *Assume that as $t \rightarrow 0+$, $f(t) \rightarrow f(0)$, where $f(0) \in (0, \infty)$. Then as $n \rightarrow \infty$,*

$$\frac{T_n^{NN}}{\log n} \xrightarrow{\mathbb{P}} \frac{1}{f(0)} \quad (2.2)$$

and

$$\mathbb{E}[T_n^{NN}] \sim \frac{1}{f(0)} \log n. \quad (2.3)$$

Moreover, convergence in (2.2) happens in \mathcal{L}_2 .

When distribution F is Exponential, then the expected value of the length of NN tour among n cities, scales as $\log n$. This is a special case of Theorem 2.2, when $f(0) = 1$. The following corollary is a consequence of Theorem 2.2.

Corollary 2.1. *Let in mean field TSP, F be Exponential with mean one. Then $T_n^{NN} - \log n$ converges weakly and in \mathcal{L}_2 .*

Theorem 2.3. *Assume that as $t \rightarrow 0+$, $\frac{f(t)}{t^\alpha} \rightarrow C$, where $C > 0$ is constant and $0 < \alpha < 1$. Then as $n \rightarrow \infty$,*

$$\frac{T_n^{NN}}{n^{1-\frac{1}{1+\alpha}}} \xrightarrow{\mathbb{P}} C(\alpha) \quad (2.4)$$

where

$$C(\alpha) := \left(\frac{1+\alpha}{C}\right)^{\frac{1}{1+\alpha}} \frac{1+\alpha}{\alpha} \Gamma\left(1 + \frac{1}{1+\alpha}\right)$$

and

$$\mathbb{E}[T_n^{NN}] \sim C(\alpha) n^{1-\frac{1}{1+\alpha}} \quad (2.5)$$

Moreover, convergence in (2.4) happens in \mathcal{L}_2 .

Theorem 2.4. *Assume that as $t \rightarrow 0+$, $\frac{f(t)}{t^\alpha} \rightarrow C$, where $C > 0$ is constant and $-1 < \alpha < 0$, then the sequence $\{\mathbb{E}[T_n^{NN}]\}_{n \geq 1}$, is a convergent sequence and T_n^{NN} converges weakly and in \mathcal{L}_2 .*

Remark 2.1. Our results cover the cases where $|\alpha| < 1$. Note that the case $\alpha \leq -1$ can not happen, since f is a density function. For $\alpha \geq 1$ we did not get result in general but for the particular choice of F , namely when F is Weibull distribution with shape parameter $(1 + \alpha)$ and scale parameter 1, one can show by proper scaling, the weak limit distribution of T_n^{NN} is Normal.

The rest of the paper is structured as follows. In Section 3, we will discuss about the last edge of NN tour in the mean field TSP. Proofs of main results are in Section 4. Section 5 provides some technical results which we use in the proof of main results. Section 6 includes the discussion about the assumptions on distribution F .

3 The last edge of the NN tour

Let the distances between cities be denoted by $\{(L_{ij})_{i < j \leq n}\}_{1 \leq i \leq n-1}$ which are i.i.d with distribution F and density f . Let L_n^{last} be the length of the last edge, which joins the last visited city to the first city. Then, the length of NN tour, T_n^{NN} , can be written as

$$T_n^{NN} \stackrel{d}{=} \sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij} + L_n^{last} \quad (3.1)$$

Following proposition shows the last edge in NN tour, does not play an important role.

Proposition 3.1. *In the NN tour for mean field TSP, length of the last edge, namely L_n^{last} is bounded in probability and in fact $\mathbb{E}[L_n^{last}] \rightarrow \mu$, where μ is the mean of F . Moreover, $L_n^{last} - \mathbb{E}[L_n^{last}]$ converges in \mathcal{L}_2 .*

Since from equation (3.1) we get

$$T_n^{NN} - \mathbb{E}[T_n^{NN}] \stackrel{d}{=} \sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij} - \mathbb{E}[\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij}] + L_n^{last} - \mathbb{E}[L_n^{last}], \quad (3.2)$$

therefore if

$$\left\{ \sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij} - \mathbb{E}[\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij}] \right\}_{n > 1}$$

converges in \mathcal{L}_2 , then by Proposition 3.1,

$$\{T_n^{NN} - \mathbb{E}[T_n^{NN}]\}_{n > 1}$$

also converges in \mathcal{L}_2 .

Proof of Proposition 3.1. Let for $k = 2, 3, \dots, n-1$, $X_k := L_{1k}$ and let $X_{(k)}$ be the k^{th} order statistic of X_2, X_3, \dots, X_{n-1} . Note that by assumption X_k 's are i.i.d. F . Notice that by construction the successive vertices $1 = v_1, v_2, v_3, \dots, v_n$, the tour has the property that for every $2 \leq k \leq n$ given $\{v_2, v_3, \dots, v_{k-1}\}$ the vertex v_k is uniformly distributed on the set $\{1, 2, \dots, n\} \setminus \{1, v_2, v_3, \dots, v_{k-1}\}$. Thus for every $3 \leq k \leq n$ given v_2 , the vertex v_k is uniformly distributed on the set $\{2, 3, \dots, n\} \setminus \{v_2\}$. So in particular the last vertex of the tour v_n is also uniformly distributed on the set $\{2, 3, \dots, n\} \setminus \{v_2\}$. Hence given X_2, X_3, \dots, X_{n-1} , the length of the last edge is uniformly on $\{X_{(2)}, X_{(3)}, \dots, X_{(n-1)}\}$.

$$\begin{aligned} L_n^{last} &\stackrel{d}{=} \frac{1}{n-2} \sum_{k=2}^{n-1} X_{(k)} \\ &\stackrel{d}{=} \frac{1}{n-2} \sum_{k=1}^{n-1} X_{(k)} - \frac{X_{(1)}}{n-2} \\ &\stackrel{d}{=} \frac{1}{n-2} \sum_{k=1}^{n-1} X_k - \frac{X_{(1)}}{n-2}. \end{aligned}$$

By SLLN, $\frac{1}{n-2} \sum_{k=1}^{n-1} X_k$ converges *a.s.* to μ . Also $\frac{X_{(1)}}{n-2}$ converges *a.s.* to zero, thus

$$L_n^{last} = O_p(1).$$

Further it also shows that $\mathbb{E}[L_n^{last}] \rightarrow \mu$ as $0 \leq \mathbb{E}[X_{(1)}] \leq \mu$. To show $L_n^{last} - \mathbb{E}[L_n^{last}]$ converges in \mathcal{L}_2 , observe that

$$\begin{aligned} \mathbb{E} \left[L_n^{last} - \mathbb{E}[L_n^{last}] \right]^2 &= \text{Var} \left(\frac{1}{n-2} \sum_{k=1}^{n-1} X_k - \frac{X_{(1)}}{n-2} \right) \\ &= \frac{1}{(n-2)^2} \sum_{k=1}^{n-1} \text{Var}(X_k) + \frac{\text{Var}(X_{(1)})}{(n-2)^2} - \frac{2}{(n-2)^2} \text{Cov} \left(\sum_{k=1}^{n-1} X_k, X_{(1)} \right) \\ &= \frac{(n-1)\sigma^2}{(n-2)^2} + \frac{\text{Var}(X_{(1)})}{(n-2)^2} - \frac{2 \sum_{k=1}^{n-1} \text{Cov}(X_k, X_{(1)})}{(n-2)^2}. \end{aligned}$$

Now since $0 \leq \mathbb{E}[X_{(1)}^2] \leq \mu^2 + \sigma^2$ where σ^2 is the variance of F and $0 \leq \mathbb{E}[X_{(1)}] \leq \mu$, therefore for large n we get

$$\frac{\text{Var}(X_{(1)})}{(n-2)^2} \rightarrow 0.$$

Also $\sum_{k=1}^{n-1} \text{Cov}(X_k, X_{(1)}) = \sum_{k=1}^{n-1} (\mathbb{E}[X_k X_{(1)}] - \mathbb{E}[X_k] \mathbb{E}[X_{(1)}])$. But

$$\sum_{k=1}^{n-1} \mathbb{E}[X_k X_{(1)}] \leq (n-1)(\mu^2 + \sigma^2).$$

Therefore as $n \rightarrow \infty$,

$$\mathbb{E} \left[L_n^{last} - \mathbb{E}[L_n^{last}] \right]^2 \rightarrow 0$$

□

4 Proofs of the main results

For $1 \leq i \leq n-1$, define

$$W_i := F^{-1} \left(1 - \exp\left(-\frac{Y_i}{i}\right) \right) \quad (4.1)$$

where $\{Y_i\}_{1 \leq i \leq n-1}$ are i.i.d. Exponential with mean one. Lemma 5.2 in Section 5, shows that

$$\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij} \stackrel{d}{=} \sum_{i=1}^{n-1} W_i. \quad (4.2)$$

Therefore equation (3.1) can be rewritten as follows:

$$T_n^{NN} \stackrel{d}{=} \sum_{i=1}^{n-1} W_i + L_n^{last}. \quad (4.3)$$

In our proofs, mostly we study the properties of W_i instead of $\min_{i < j \leq n} L_{ij}$. Observe that:

$$\mathbb{P}(W_i \leq w) = 1 - \{1 - F(w)\}^i \text{ for } w \geq 0. \quad (4.4)$$

4.1 Proof of Theorem 2.1

To show $\{T_n^{NN} - \mathbb{E}[T_n^{NN}]\}_{n \geq 1}$ converges in \mathcal{L}_2 , by Proposition 3.1 and equation (4.3), it is enough to show $\{\sum_{i=1}^{n-1} (W_i - \mathbb{E}[W_i])\}_{n \geq 1}$ converges in \mathcal{L}_2 .

Recall as $t \rightarrow 0+$, by assumption $\frac{f(t)}{t^\alpha} \rightarrow C$, therefore given $\epsilon > 0$, there exists $\delta > 0$, such that for all $0 < t < \delta$, we have

$$(C - \epsilon)t^\alpha < f(t) < (C + \epsilon)t^\alpha.$$

Hence for $0 < x < \delta$,

$$\frac{(C - \epsilon)}{1 + \alpha} x^{1+\alpha} < F(x) < \frac{(C + \epsilon)}{1 + \alpha} x^{1+\alpha}$$

which implies

$$\left(\frac{1 + \alpha}{C + \epsilon}\right)^{\frac{1}{1+\alpha}} x^{\frac{1}{1+\alpha}} < F^{-1}(x) < \left(\frac{1 + \alpha}{C - \epsilon}\right)^{\frac{1}{1+\alpha}} x^{\frac{1}{1+\alpha}}. \quad (4.5)$$

Put $\delta_1 := -\ln(1 - \delta)$. Hence if $\frac{Y_i}{i} < \delta_1$ (that ensures $1 - \exp(-\frac{Y_i}{i}) < \delta$), then we get

$$W_i \mathbf{1} \left[\frac{Y_i}{i} < \delta_1 \right] < \left(\frac{1 + \alpha}{C - \epsilon}\right)^{\frac{1}{1+\alpha}} \left(1 - \exp(-\frac{Y_i}{i})\right)^{\frac{1}{1+\alpha}} \mathbf{1} \left[\frac{Y_i}{i} < \delta_1 \right]. \quad (4.6)$$

Observe that for $\beta > 0$,

$$\begin{aligned} \mathbb{E} \left[\left(1 - \exp(-\frac{Y_i}{i})\right)^\beta \right] &= \int_0^\infty (1 - \exp(-y/i))^\beta \exp(-y) dy \\ &= i \int_0^1 u^\beta (1 - u)^{i-1} du \\ &= \Gamma(1 + \beta) \frac{\Gamma(i + 1)}{\Gamma(i + 1 + \beta)} \\ &\leq \Gamma(2 + \beta) \frac{1}{(i + 1 + \beta)^\beta}. \end{aligned}$$

The last inequality follows from the Wendel's double inequality [8], which says for real $x > 0$ and $0 < s < 1$ we have

$$\frac{x}{(x + s)^{1-s}} \Gamma(x) \leq \Gamma(x + s) \leq x^s \Gamma(x) \quad (4.7)$$

Therefore

$$\mathbb{E} \left[W_i^2 \mathbf{1} \left[\frac{Y_i}{i} < \delta_1 \right] \right] < \left(\frac{1 + \alpha}{C - \epsilon}\right)^{\frac{2}{1+\alpha}} \Gamma \left(2 + \frac{2}{1 + \alpha} \right) \frac{1}{\left(i + 1 + \frac{2}{1+\alpha}\right)^{\frac{2}{1+\alpha}}}. \quad (4.8)$$

Now as $i \rightarrow \infty$, $\frac{Y_i}{i} \xrightarrow{a.s.} 0$. Define

$$I_0(\omega) := \min \left\{ i \mid \frac{Y_j(\omega)}{j} < \delta_1, \forall j \geq i \right\}. \quad (4.9)$$

Fix $m > 1$, then

$$[I_0 = m] = \left[\frac{Y_i}{i} < \delta_1, \forall i \geq m \quad \text{and} \quad \frac{Y_{m-1}}{m-1} > \delta_1 \right].$$

Hence

$$\mathbb{P}(I_0 = m) \leq e^{-(m-1)\delta_1}$$

Now,

$$\sum_{i=1}^{\infty} \mathbb{E}[W_i^2] = \sum_{m=1}^{\infty} \mathbb{E}\left[\sum_{i=1}^{m-1} W_i^2 \mathbf{1}(I_0 = m)\right] + \sum_{m=1}^{\infty} \mathbb{E}\left[\sum_{i=m}^{\infty} W_i^2 \mathbf{1}(I_0 = m)\right].$$

But,

$$\begin{aligned} \mathbb{E}\left[\sum_{i=1}^{m-1} W_i^2 \mathbf{1}(I_0 = m)\right] &= \mathbb{E}\left[\sum_{i=1}^{m-2} W_i^2 \mathbf{1}(I_0 = m)\right] + \mathbb{E}[W_{m-1}^2 \mathbf{1}(I_0 = m)] \\ &\leq e^{-(m-1)\delta_1} \sum_{i=1}^{m-2} \mathbb{E}[W_i^2] + \mathbb{E}[W_{m-1}^2 \mathbf{1}(W_{m-1} > F^{-1}(1 - e^{-\delta_1}))] \end{aligned} \quad (4.10)$$

Now,

$$\sum_{i=1}^{m-2} \mathbb{E}[W_i^2] = (m-2)\mathbb{E}[W_1^2], \quad (4.11)$$

and

$$\begin{aligned} &\mathbb{E}[W_{m-1}^2 \mathbf{1}(W_{m-1} > F^{-1}(1 - e^{-\delta_1}))] \\ &= 2 \int_{F^{-1}(1 - e^{-\delta_1})}^{\infty} w \mathbb{P}(W_{m-1} > w) dw \\ &= 2e^{-(m-1)\delta_1} \int_{F^{-1}(1 - e^{-\delta_1})}^{\infty} w \mathbb{P}(W_{m-1} > w | W_{m-1} > F^{-1}(1 - e^{-\delta_1})) dw \\ &= 2e^{-(m-1)\delta_1} \int_{F^{-1}(1 - e^{-\delta_1})}^{\infty} w \mathbb{P}\left(W_{m-1} > w - F^{-1}(1 - e^{-\delta_1})\right) dw \\ &= 2e^{-(m-1)\delta_1} \int_0^{\infty} [t + F^{-1}(1 - e^{-\delta_1})] \mathbb{P}(W_{m-1} > t) dt \\ &= 2e^{-(m-1)\delta_1} \int_0^{\infty} [t + F^{-1}(1 - e^{-\delta_1})] [1 - F(t)]^{m-1} dt \end{aligned}$$

Third equality is because of the memoryless property of the Exponential distribution. Using Lemma 6.1 and equation (4.11), we get

$$\sum_{m=1}^{\infty} \mathbb{E}\left[\sum_{i=1}^{m-1} W_i^2 \mathbf{1}(I_0 = m)\right] < \infty \quad (4.12)$$

Now by assumption since $|\alpha| < 1$, we have $\frac{2}{1+\alpha} > 1$, therefore for $i \geq m$ by (4.8) we get

$$\mathbb{E}\left[\sum_{i=m}^{\infty} W_i^2 \mathbf{1}(I_0 = m)\right] < K e^{-(m-1)\delta_1} \quad (4.13)$$

where K is constant. Hence from equations (4.12) and (4.13) we conclude

$$\sum_{i=1}^{\infty} \mathbb{E}[W_i^2] < \infty \quad (4.14)$$

Therefore $\text{Var}[\sum_{i=1}^n W_i]$ is bounded for all n . This shows $\sum_{i=1}^{n-1} (W_i - \mathbb{E}[W_i])$ as a Martingale converges *a.s.* and in \mathcal{L}_2 . Hence by Proposition 3.1 and equation (4.3), we conclude that $T_n^{NN} - \mathbb{E}[T_n^{NN}]$ converges in \mathcal{L}_2 . \square

4.2 Proof of Theorem 2.2

We will show

$$\frac{T_n^{NN}}{\log n} \xrightarrow{\mathcal{L}_2} \frac{1}{f(0)} \quad \text{as } n \rightarrow \infty,$$

which will imply equation (2.2). Note that we have,

$$\begin{aligned} \mathbb{E} \left[\frac{T_n^{NN}}{\log n} - \frac{1}{f(0)} \right]^2 &= \mathbb{E} \left[\frac{T_n^{NN} - \mathbb{E}[T_n^{NN}]}{\log n} + \frac{\mathbb{E}[T_n^{NN}]}{\log n} - \frac{1}{f(0)} \right]^2 \\ &= \mathbb{E} \left[\frac{T_n^{NN} - \mathbb{E}[T_n^{NN}]}{\log n} \right]^2 + \left[\frac{\mathbb{E}[T_n^{NN}]}{\log n} - \frac{1}{f(0)} \right]^2. \end{aligned} \quad (4.15)$$

But, by Theorem 2.1 (\mathcal{L}_2 convergence of $T_n^{NN} - \mathbb{E}[T_n^{NN}]$), first term in (4.15) converges to zero as $n \rightarrow \infty$. Convergence to zero of the second term in (4.15) follows from the following observation. By assumption as $i \rightarrow \infty$,

$$\frac{f(0)W_i}{\frac{Y_i}{i}} \rightarrow 1 \quad a.s.$$

where Y_i 's are i.i.d. Exponential with mean one and $W_i = F^{-1} \left(1 - \exp(-\frac{Y_i}{i}) \right)$. Therefore as $n \rightarrow \infty$

$$\frac{f(0) \sum_{i=1}^{n-1} W_i}{\sum_{i=1}^{n-1} \frac{Y_i}{i}} \rightarrow 1 \quad a.s.$$

Now, since $\text{Var} \left[\sum_{i=1}^{n-1} \frac{Y_i}{i} \right] < \infty$ for all n , therefore by Martingale convergence theorem almost surely $\sum_{i=1}^{n-1} \frac{Y_i}{i} - \mathbb{E} \left[\sum_{i=1}^{n-1} \frac{Y_i}{i} \right]$ converges. But $\mathbb{E} \left[\sum_{i=1}^{n-1} \frac{Y_i}{i} \right] = \sum_{i=1}^n \frac{1}{i} \sim \log n$, thus

$$\frac{f(0) \sum_{i=1}^{n-1} W_i}{\log n} \rightarrow 1 \quad a.s. \quad (4.16)$$

As we saw in the proof of Theorem 2.1, $\sum_{i=1}^{n-1} W_i - \mathbb{E}[\sum_{i=1}^{n-1} W_i]$ converges *a.s.* to a random variable. This observation along with (4.16) give

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\sum_{i=1}^{n-1} W_i]}{\log n} = \frac{1}{f(0)} \quad (4.17)$$

and therefore by equation (4.3) and Proposition 3.1,

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[T_n^{NN}]}{\log n} = \frac{1}{f(0)}$$

This also proved $\mathbb{E}[T_n^{NN}] \sim \frac{1}{f(0)} \log n$. \square

4.3 Proof of Corollary 2.1

Consider a mean field TSP on n cities $\{1, 2, \dots, n\}$, where for each $1 \leq i \leq n-1$, the intercity distances $\{L_{ij}\}_{i < j \leq n}$, are i.i.d. Exponential with mean one. Starting at city 1, our job is to find the nearest city to it, that means to find

$$\min_{1 < j \leq n} L_{1j}$$

Now we have a tour, with 2 cities in it. Finding the next nearest city to last visited city in this tour, in distribution is the same as finding the *minimum* of $n-3$ independent Exponential random variables. Keep visiting nearest unvisited city to the last visited city in tour. Since $\min_{i < j \leq n} L_{ij}$ has Exponential distribution with mean $\frac{1}{n-i}$, then we get

$$\mathbb{E}[\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij}] = \frac{1}{n-1} + \frac{1}{n-2} + \dots + \frac{1}{2} + 1 \quad (4.18)$$

Since $\text{Var}[\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij}] = \sum_{i=1}^{n-1} \frac{1}{i^2}$, therefore for all $n \geq 1$, $\text{Var}\left(\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij} - \mathbb{E}[\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij}]\right)$ is bounded. Hence by Martingale convergence theorem, we conclude that the Martingale sequence

$$\left\{ \sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij} - \mathbb{E}[\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij}] \right\}_{n \geq 1} \text{ converges } a.s. \quad (4.19)$$

Note that as we saw in (4.18), $\mathbb{E}[\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij}] = \sum_{i=1}^{n-1} \frac{1}{i}$. Using the fact that,

$$\sum_{i=1}^n \frac{1}{i} = \log n + \gamma + O\left(\frac{1}{n}\right)$$

where $\gamma := \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right)$ is the Euler constant, shows that $\left\{ \mathbb{E}[\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij}] - \log n \right\}_{n \geq 1}$ is a convergent sequence. Now since

$$\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij} - \log n = \sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij} - \mathbb{E}[\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij}] + \mathbb{E}[\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij}] - \log n$$

thus from (4.19) and equation (3.1) we get $(T_n^{NN} - \log n)_{n \geq 1}$ converges weakly as well as in \mathcal{L}_2 . \square

4.4 Proof of Theorem 2.3

Recall the inequality (4.5) in the proof of Theorem 2.1. Therefore by the assumption of the theorem and (4.5), as $i \rightarrow \infty$,

$$\frac{(\frac{C}{1+\alpha})^{\frac{1}{1+\alpha}} W_i}{(\frac{Y_i}{i})^{\frac{1}{1+\alpha}}} \rightarrow 1 \quad a.s.$$

where Y_i 's are i.i.d. Exponential with mean one and $W_i = F^{-1} \left(1 - \exp(-\frac{Y_i}{i}) \right)$. Therefore as $n \rightarrow \infty$

$$\frac{(\frac{C}{1+\alpha})^{\frac{1}{1+\alpha}} \sum_{i=1}^{n-1} W_i}{\sum_{i=1}^{n-1} (\frac{Y_i}{i})^{\frac{1}{1+\alpha}}} \rightarrow 1 \quad a.s.$$

Since $0 < \alpha < 1$ so $\frac{2}{1+\alpha} > 1$, thus $\text{Var} \left(\sum_{i=1}^{n-1} (\frac{Y_i}{i})^{\frac{1}{1+\alpha}} \right) < \infty$ for all n , therefore by Martingale convergence theorem almost surely $\sum_{i=1}^{n-1} (\frac{Y_i}{i})^{\frac{1}{1+\alpha}} - \mathbb{E} \left[\sum_{i=1}^{n-1} (\frac{Y_i}{i})^{\frac{1}{1+\alpha}} \right]$ converges. But

$$\mathbb{E} \left[\sum_{i=1}^{n-1} (\frac{Y_i}{i})^{\frac{1}{1+\alpha}} \right] = \Gamma(1 + \frac{1}{1+\alpha}) \sum_{i=1}^{n-1} (\frac{1}{i})^{\frac{1}{1+\alpha}}.$$

Thus

$$\frac{\sum_{i=1}^{n-1} W_i}{C(\alpha) n^{1-\frac{1}{1+\alpha}}} \rightarrow 1 \quad a.s. \quad (4.20)$$

where

$$C(\alpha) := (\frac{1+\alpha}{C})^{\frac{1}{1+\alpha}} \frac{1+\alpha}{\alpha} \Gamma(1 + \frac{1}{1+\alpha})$$

Now

$$\sum_{i=1}^{n-1} W_i - C(\alpha) n^{1-\frac{1}{1+\alpha}} = \sum_{i=1}^{n-1} W_i - \mathbb{E}[\sum_{i=1}^{n-1} W_i] + \mathbb{E}[\sum_{i=1}^{n-1} W_i] - C(\alpha) n^{1-\frac{1}{1+\alpha}}$$

Recall that by Theorem 2.1, $\sum_{i=1}^{n-1} W_i - \mathbb{E}[\sum_{i=1}^{n-1} W_i]$ has a almost sure limit, so using (4.20) we get

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[\sum_{i=1}^{n-1} W_i]}{n^{1-\frac{1}{1+\alpha}}} = C(\alpha) \quad (4.21)$$

and hence by equation (4.3),

$$\mathbb{E}[T_n^{NN}] \sim C(\alpha) n^{1-\frac{1}{1+\alpha}}$$

This along with the Theorem 2.1 shows

$$\mathbb{E}\left[\frac{T_n^{NN}}{n^{1-\frac{1}{1+\alpha}}} - C(\alpha)\right]^2 = \mathbb{E}\left[\frac{T_n^{NN} - \mathbb{E}[T_n^{NN}]}{n^{1-\frac{1}{1+\alpha}}} + \frac{\mathbb{E}[T_n^{NN}]}{n^{1-\frac{1}{1+\alpha}}} - C(\alpha)\right]^2$$

converges to zero as $n \rightarrow \infty$. Hence

$$\frac{T_n^{NN}}{n^{1-\frac{1}{1+\alpha}}} \xrightarrow{\mathbb{P}} C(\alpha).$$

□

4.5 Proof of Theorem 2.4

As it has mentioned in the proof of Theorem 2.1, since $\frac{1}{1+\alpha} > 1$, we get

$$\sup_{n \geq 1} \text{Var}\left(\sum_{i=1}^{n-1} W_i\right) < \infty.$$

Therefore $\sum_{i=1}^{n-1} W_i - \mathbb{E}\left[\sum_{i=1}^{n-1} W_i\right]$ as a martingale converges *a.s.* and in \mathcal{L}_2 . So by equation (4.3) and Proposition 3.1, $T_n^{NN} - \mathbb{E}[T_n^{NN}]$ converges in \mathcal{L}_2 . Now from Lemma 5.1 we get

$$\mathbb{E}[T_n^{NN}] = \int_0^1 \frac{(1-w)(1-[1-w]^{n-1})}{w} \frac{1}{f(F^{-1}(w))} dw + \mathbb{E}[L_n^{last}]. \quad (4.22)$$

Now using equation (4.5) we get that there is a constant c which may depends on α such that for every $n \geq 1$,

$$\int_0^1 \frac{(1-w)(1-[1-w]^{n-1})}{w} \frac{1}{f(F^{-1}(w))} dw \leq c \int_0^1 \frac{1}{w^{1+\frac{\alpha}{1+\alpha}}} dw < \infty \quad (4.23)$$

where the last inequality follows since $-1 < \alpha < 0$. Thus using DCT we get that the first term in the right hand side of the equation (4.22) converges and hence from Proposition 3.1 we conclude that $(\mathbb{E}[T_n^{NN}])_{n \geq 1}$ also converges. Therefore $(T_n^{NN})_{n \geq 1}$ converges in \mathcal{L}_2 . □

5 Technical result

We first state a lemma which expands $\mathbb{E}[T_n^{NN}]$ in terms of Uniform random variables, under certain assumption on F . This lemma shows that how behavior of T_n^{NN} depends on the density f near zero.

Lemma 5.1. *Let $(L_{ij})_{i < j \leq n}$ for $i = 1, \dots, n-1$ be i.i.d F supported on $[0, \infty)$. Assume that $\forall \epsilon > 0, F([0, \epsilon)) > 0$. Let f be the density of F . Then*

$$\mathbb{E}[T_n^{NN}] = \int_0^1 \frac{(1-w)(1-[1-w]^{n-1})}{w} \frac{1}{f(F^{-1}(w))} dw + \mathbb{E}[L_n^{last}]$$

Proof. Let $\bar{F}(t) = 1 - F(t)$. From the equation (3.1) We have

$$\mathbb{E}[T_n^{NN}] = \sum_{i=1}^{n-1} \mathbb{E}[\min_{i < j \leq n} L_{ij}] + \mathbb{E}[L_n^{last}]$$

But,

$$\mathbb{E}[\min_{i < j \leq n} L_{ij}] = \int_0^\infty [\bar{F}(t)]^{n-i} dt,$$

hence,

$$\mathbb{E}[T_n^{NN}] = \int_0^\infty \frac{\bar{F}(t)(1 - [\bar{F}(t)]^{n-1})}{F(t)} dt + \mathbb{E}[L_n^{last}]$$

Now by assumption, density f exists. Put $w = F(t)$, then we have $t = F^{-1}(w)$, $dw = f(t)dt$. Replace $F(t)$ by w inside integral to get the result. \square

Our next lemma gives the derivation of the equation (4.2)

Lemma 5.2. *Let the distances between cities, $(L_{ij})_{i < j \leq n}$ for $i = 1, \dots, n$ be i.i.d F . Define $W_i := F^{-1}\left(1 - \exp(-\frac{Y_i}{i})\right)$ where $\{Y_i\}_{1 \leq i \leq n-1}$ are i.i.d. Exponential with mean one. Then*

$$\sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij} \stackrel{d}{=} \sum_{i=1}^{n-1} W_i$$

Proof. Let $(\xi_{ij})_{i < j \leq n}$ be i.i.d. Exponential with mean one. Then

$$\begin{aligned} \sum_{i=1}^{n-1} \min_{i < j \leq n} L_{ij} &\stackrel{d}{=} \sum_{i=1}^{n-1} \min_{i < j \leq n} F^{-1}(1 - e^{-\xi_{ij}}) \\ &\stackrel{d}{=} \sum_{i=1}^{n-1} F^{-1}(1 - e^{-\min_{i < j \leq n} \xi_{ij}}) \\ &\stackrel{d}{=} \sum_{i=1}^{n-1} F^{-1}(1 - e^{-\frac{Y_i}{i}}) \end{aligned}$$

where Y_i 's are i.i.d. Exponential with mean one. \square

6 Discussion

In our theorems, we assumed that the second moment of F exists. In general, suppose nonnegative and independent random variables Z_1, Z_2, \dots are from distribution F such that for some $\beta > 0$, $\mathbb{E}[Z_1^\beta] < \infty$.

Lemma 6.1. *Suppose random variable $Z \geq 0$ and for some $\beta > 0$, $\mathbb{E}[Z^\beta] < \infty$. Then for $k > \frac{2}{\beta}$*

$$\int_0^\infty t \{\mathbb{P}(Z > t)\}^k dt < \infty$$

The proof of this lemma, follows easily by using Markov's inequality. Now as before let $W_i = F^{-1}\left(1 - \exp(-\frac{Y_i}{i})\right)$ where Y_i 's are Exponential with mean one. Put $k = \left\lfloor \frac{2}{\beta} \right\rfloor + 1$. Assume that as $t \rightarrow 0+$, $\frac{f(t)}{t^\alpha} \rightarrow C$, where $C \in (0, \infty)$ is constant and $-1 < \alpha < 1$. Then as $n \rightarrow \infty$, we have

$$\sum_{i=k}^{\infty} \mathbb{E}[W_i^2] < \infty.$$

Therefore

$$\sum_{i=k}^{\infty} \text{Var}(W_i) < \infty,$$

and hence $\sum_{i=k}^{n-1} (W_i - \mathbb{E}[W_i])$ converges almost surly. Now since

$$T_n^{NN} \stackrel{d}{=} \sum_{i=1}^{n-1} W_i + L_n^{last},$$

hence by Proposition 3.1, $T_n^{NN} - \mathbb{E}[T_n^{NN}]$ converges weakly. Thus except those on \mathcal{L}_2 convergence all the results stated in Section 2 holds.

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